What is a four dimensional space like?

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We have already seen that there is nothing terribly mysterious about adding one dimension to space to form a spacetime. Nonetheless it is hard to resist a lingering uneasiness about the idea of a four dimensional spacetime. The problem is not the *time* part of a four dimensional spacetime; it is the *four*. One can readily imagine the three axes of a three dimensional space: up-down, across and back to front. But where are we to put the fourth axis to make a four dimensional space?

My present purpose is to show you that there is nothing at all mysterious in the four dimensions of a spacetime. To do this, I will drop the time part completely. I will just consider a four dimensional space; that is, a space just like our three dimensional space, but with one extra dimension. What would it be like?

With no effort whatever, I can visualize a *three* dimensional space--and you can too. What would it be like to live in a three dimensional cube? To be asked to visualize that is like being asked to breathe or blink. It is effortless. There we sit in the cube with its six square walls and eight corners. Our mind's eye lets us hover about inside.

Can I visualize what it would be like to live in the four dimensional analog of a cube, a four dimensional cube or "tesseract"? I cannot visualize this with the same effortless immediacy. I doubt that you can as well. But that is just about the only thing we cannot do. Otherwise we can determine all the properties of a tesseract and just what it would be like to live in one. There are many techniques for doing this. I will show you

Don't be confused by what I am trying to show here. I am NOT saying that our space is really four dimensional. It is not. It has only three spatial dimensions. I AM merely trying to show that we can understand what it would one below. It involves progressing through the sequence of dimensions, extrapolating the natural inferences at each step up to the fourth dimension. Once you have seen how this is done for the special case of a tesseract, you will have no trouble applying it to other cases.

be like if space did happen to have four dimensions.

The exercise is not so different from showing that we can understand what the earth would be like if it had two moons instead of one. It actually has only one moon. But we can figure out in some detail how things would be different if it had two moons.

The door to the fourth dimension is opening.

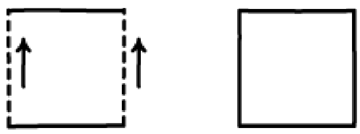
The one dimensional interval

The one dimensional analog of a cube is an interval. It is formed by taking a dimensionless point and dragging it through a distance. That distance could be 2 inches or 3 feet or anything. Let us call the distance "L".

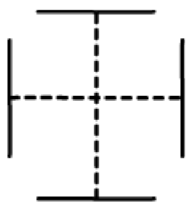
The interval has length L. It is bounded by 2 points as its faces--the two points at either end of the interval.

The two dimensional square

The two dimensional analog of a cube is a square. It is formed by dragging the one dimensional interval through a distance L in the second dimension.



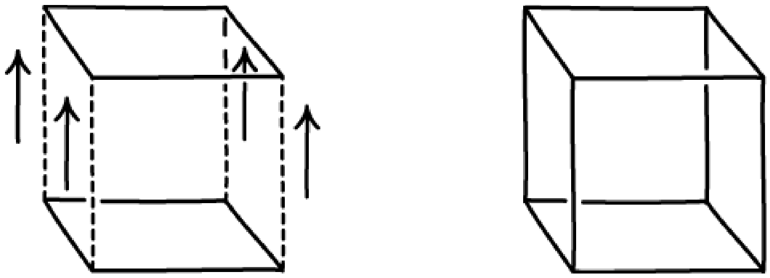
The square has area L^2. It is bounded by faces on 4 sides. The faces are intervals of length L. We know there are four of them since its two dimensional axes must be capped on either end by faces.



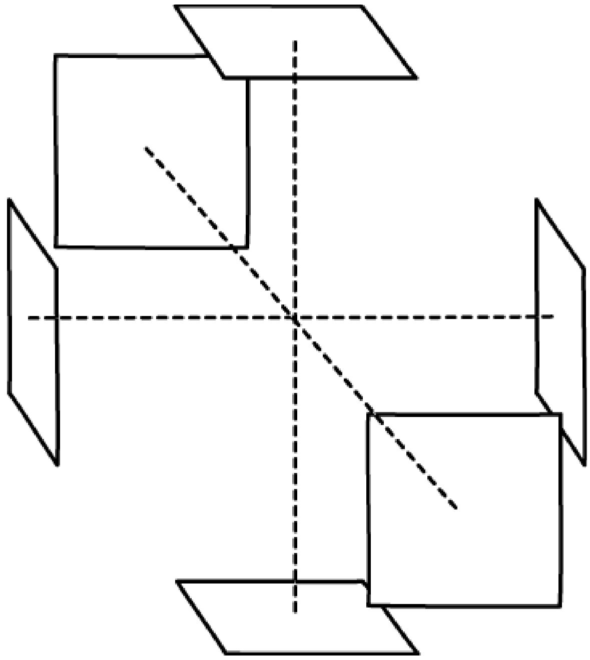
So we have 2 dimensions x 2 faces each = 4 faces. The faces together form a perimeter of 4xL in length.

The three dimensional cube

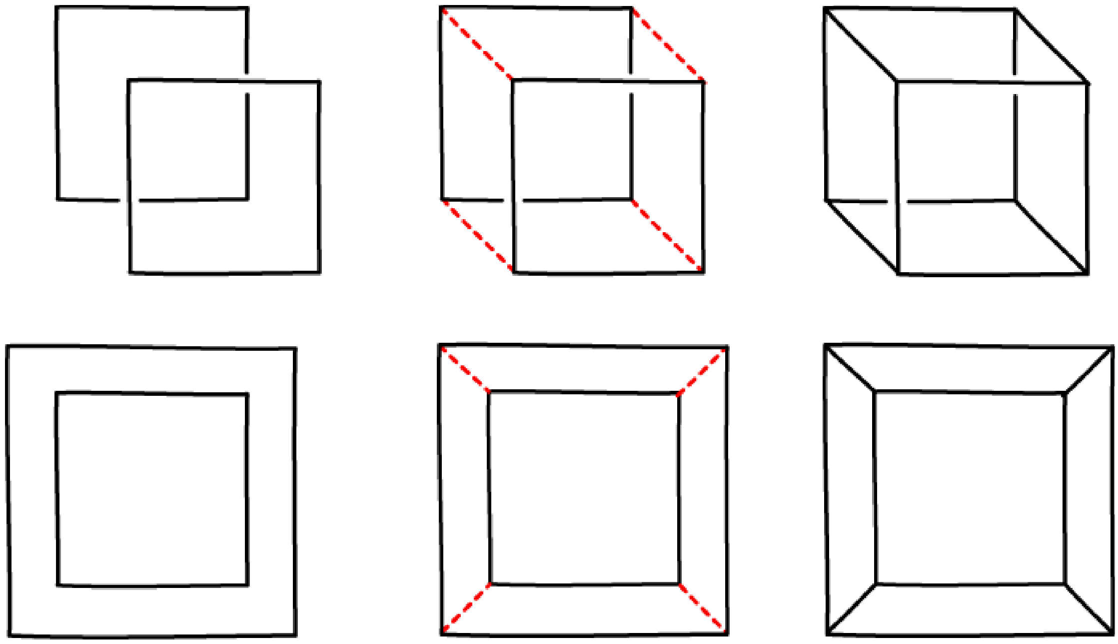
To form a cube, we take the square and drag it a distance L in the third dimension.



The cube has volume L^3. It is bounded by faces on 6 sides. The faces are squares of area L^2. We know there are 6 of them since its three dimensional axes must be capped on either end by faces.



So we have 3 dimensions x 2 faces each = 6 faces. The faces together form a surface of $6xL^2$ in area. Drawing a picture of a three dimensional cube on a two dimensional surface is equally easy. We take two of its faces--two squares--and connect the corners.

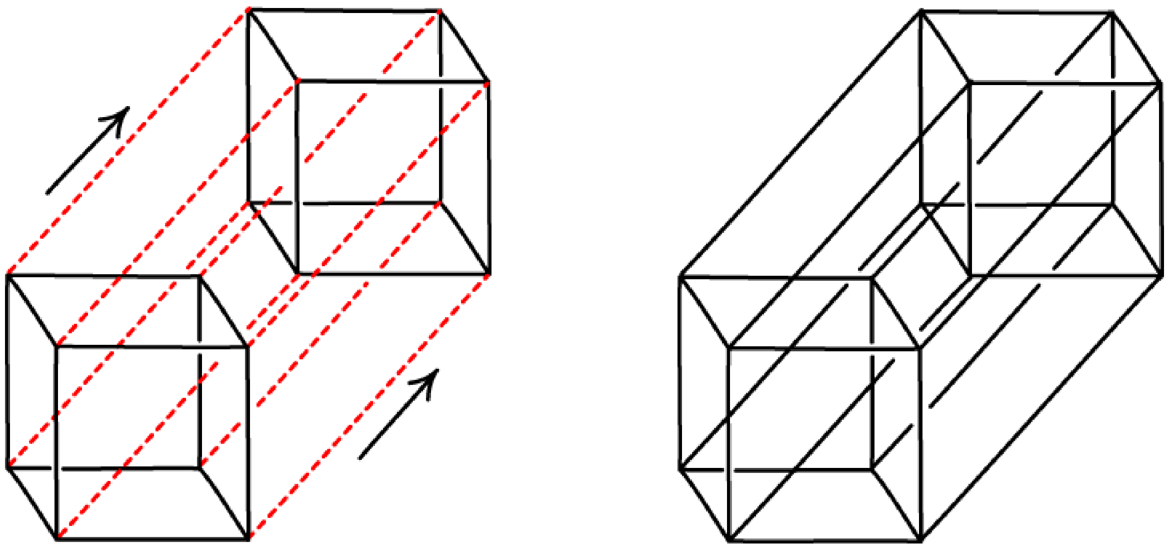


There are several ways of doing the drawing that corresponds to looking at the cube from different angles. The figure shows two ways of doing it. The first gives an oblique view; the second looks along one of the axes.

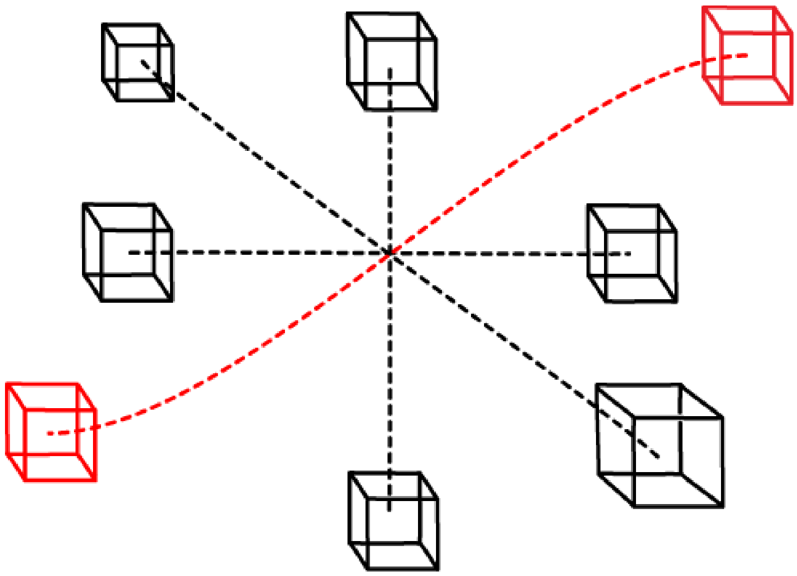
The four dimensional cube: the tesseract

So far I hope you have found our constructions entirely unchallenging. The next step into four dimensions can be done equally mechanically. We just systematically repeat every step above. The only difference is that this time we cannot readily form a mental picture of what we are building. But we can know all its properties!

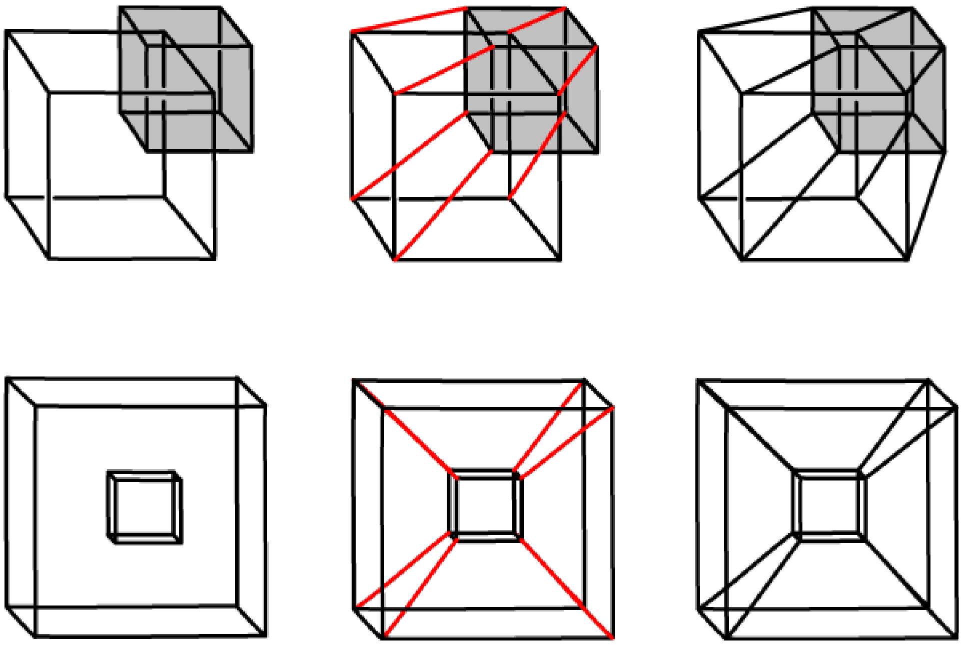
To form a tesseract, we take the cube and drag it a distance L in the *fourth* dimension. We cannot visualize exactly what that looks like, but it is something like this:



The tesseract has volume L^4. It is bounded by faces on 8 sides. The faces are cubes of volume L^3. We know there are 8 of them since its four dimensional axes must be capped on either end by faces--two cubical faces per axis. Once again, we cannot visualize all four of these capped dimenions. We can at best visualize three directions perpendicular to each other. We then somehow add in the fourth (in red):



So we have 4 dimensions x 2 faces each = 8 faces. The faces together form a "surface" (really a three dimensional volume) of $8xL^3$ in volume. Drawing a picture of a four dimensional tesseract in a three dimensional space is straightforward. We take two of its faces--two cubes--and connect the corners.



There are several ways of doing the drawing that corresponds to looking at the tesseract from different angles. The figure shows two ways of doing it. The first gives an oblique view; the second looks along one of the axes.

So now we seem to know everything there is to know about the tesseract! We know its volume in four dimensional space, how it is put together out of eight cubes as surfaces and even what the volume of its surface is ($8xL^3$).

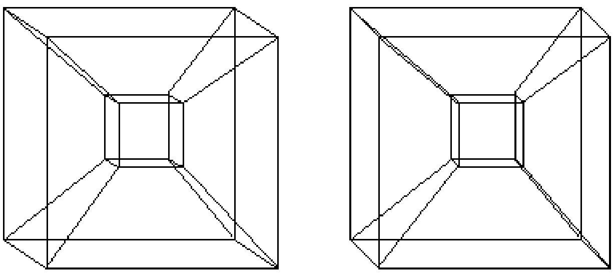
Stereovision

The "drawings" of the tesseract are hard to see clearly. That is because they are really supposed to be three dimensional models in a three dimensional space. So what we have above are two dimensional drawings of three dimensional models of a four dimensional tesseract. No wonder it is getting messy!

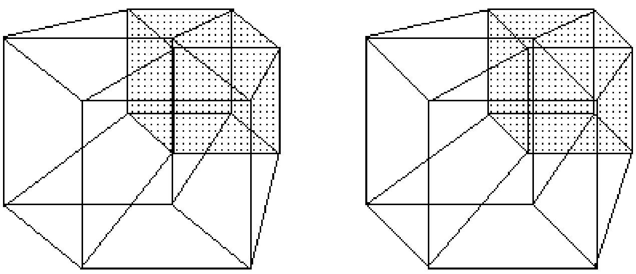
The images below are stereo pairs. If you are familiar with how to view them, you will see that they give you a nice stereo view of the three dimensional model. If these are new to you, they take practice to see. You need to relax your view until your left eye looks at the left image and the right eye looks at the right image.

But how can you learn to do this? I find it easiest to start if I sit far away from the screen and gaze out into the distance over the top of the screen. I see the two somewhat blurred images on the edge of my field of vision. As long as I don't focus on them, they start to drift together. That is the motion you want. The more they drift together the better. I try to reinforce the drift as best I can while carefully moving my view toward the images. The goal is to get the two images to merge.When they do, I keep staring at the merged images, the focus improves and the full three dimensional stereo effect snaps in sharply. The effect is striking and worth a little effort.

This pair is easier to fuse:



and this one is a little harder:



Summary table

We can summarize the development of the properties of a tesseract as follows:

Dimension	Figure	Face	Volume	Number of faces	Volume of surface/ perimeter
1	interval	point	L	1x2=2	two points
2	square	interval	L^2	2x2=4	4L
3	cube	square	L^3	3x2=6	$6L^2$
4	tesseract	cube	L^4	4x2=8	$8L^3$

A roomy challenge

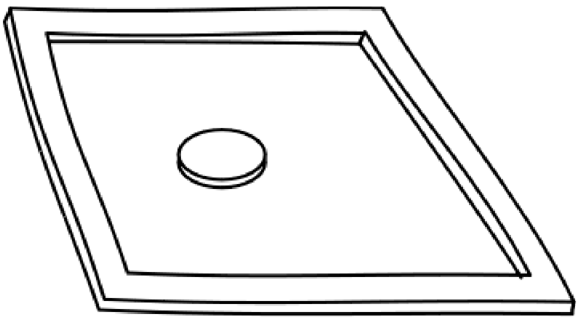
If you were to live in a tesseract, you might choose to live in its three dimensional surface, much as a two dimensional person might choose live in the 6 square rooms that form the two dimensional surface of a cube. So your house would be the eight cubes that form the surface of the tesseract. Imagine that there are doors where ever two of these cubes meet. If you are in one of these rooms, how many doors would you see? What would the next room look like if you passed through one of the doors? How many doors must you pass through to get to the farthest room? How many paths lead to that farthest room? Could you have any windows to outside the tesseract? What about windows to inside the tesseract?

Some of these questions are not easy. To answer them, go back to the easy case of a three dimensional cube with faces consisting of squares. Ask the analogous questions there and just extrapolate the answers to the tesseract.

A knotty challenge

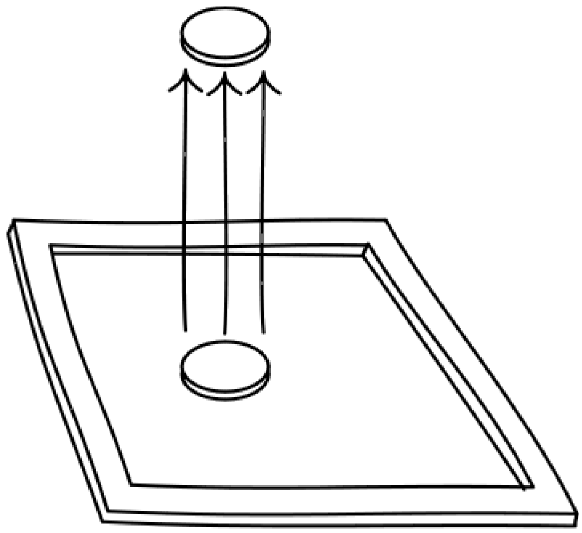
Access to a fourth dimension makes many things possible that would otherwise be quite impossible. To see how this works, we'll use the strategy of thinking out a process in a three dimensional space. Then we replicated it in a four dimensional space.

Consider a coin lying in a frame on a table top.

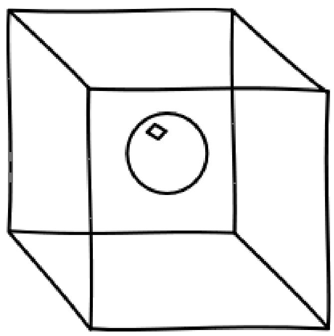


There is no way the coin can be removed from the frame within the confines of the two dimensional surface of the table. Now recall that we have access to a third dimension. The coin is easily removed merely by lifting it into the third dimension, the height above the table.

We are then free to move the coin as we please in the higher layer and then lower back to the tabletop outside the frame.



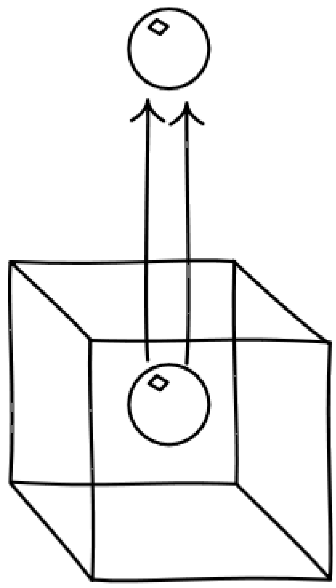
The thing to notice about the lifting is that the motion does not move the coin at all in the two horizontal directions of the two dimensional space. So the motion never brings it near the frame and there is no danger of collision with the frame.

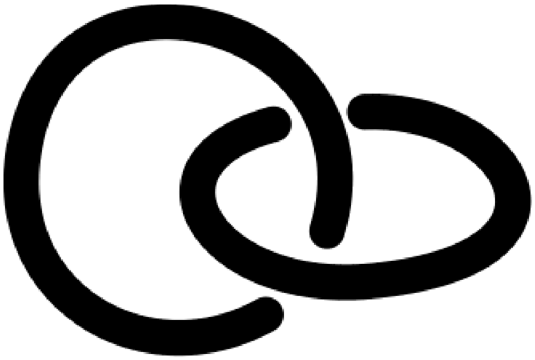


Now repeat this analysis for its analog in one higher dimension, a marble trapped within a three dimensional box.

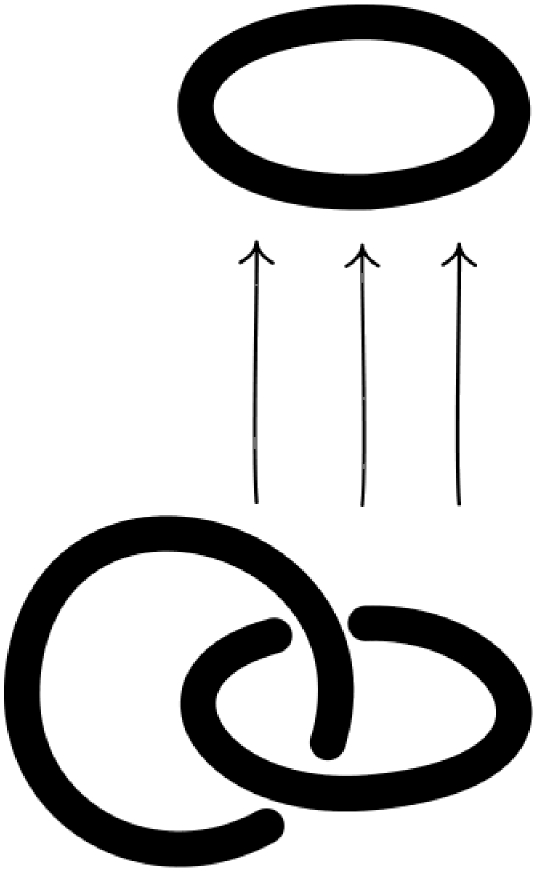
The marble can be removed in exactly the same way by "lifting" it, this time into the fourth dimension. As with the coin in the frame, the key thing to note is that in this lifting motion, the marble's position in the three spatial directions of the box are unchanged. The marble never comes near the walls and there is no danger of colliding with them.

Once it is lifted into a new three dimensional space, it can be moved around freely in that space and lowered back into the original three dimensional space, but now outside the box.





Now finally consider two linked rings in some three dimensional space. Can we separate them using access to a fourth dimension?



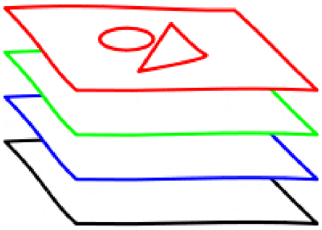
It can be done by exactly the same process of lifting one of the rings into the fourth dimension. As before, note that the lifting does not move the ring in any of the three directions of the three dimensional space holding the initially linked rings. So the motion risks no collisions of moved ring with the other. The lifting simply elevates the moved ring to a new three dimensional layer of the four dimensional space in which no part of the other ring is found. The moved ring can then be freely relocated in that new layer and, if we pleased lowered back into the original three dimensional space in quite a different location.

Now comes the knotty challenge. We are familiar in our three dimensional space with tying knots in a rope. Some knots are just apparent tangles that can come apart pretty easily. Others are real and can only be undone by threading the end of the rope through a loop. So take this to be a real knot: one that cannot be undone by any manipulation of the rope if we cannot get hold of the ends. (Imagine, if you like, that they are each anchored to a wall and cannot be removed.)

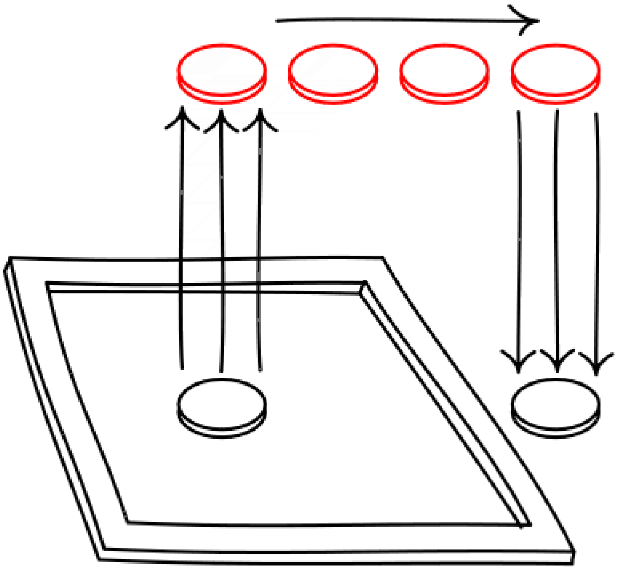
The challenge is to convince yourself that there are no real knots in ropes in a four dimensional space. The principal aid you will need is the manipulation above of the linked rings. To get yourself started, imagine how you would use a fourth dimension to untie some simple knot you can easily imagine.

Using colors to visualize the extra dimension

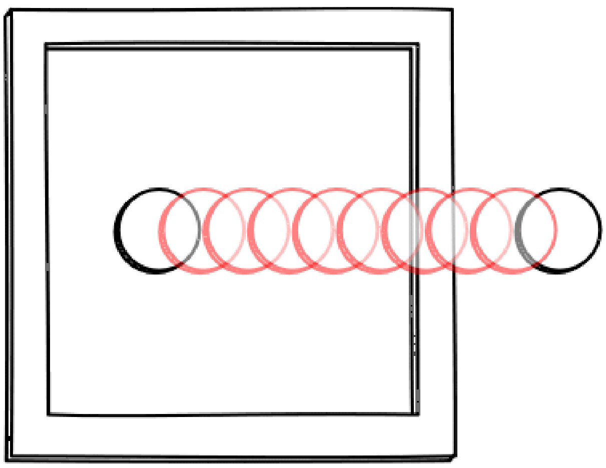
Does the general idea of "lifting" an object into the fourth dimension still seem elusive? If so, here's a technique for visualizing it that may just help. The trick is to imagine that differences in position in the extra dimension of space can be represented by differences of colors.



Here's how it works when we start with a two dimensional space and lift into the third dimension. The objects in the original two dimensional space are black. As we lift through the third dimension, they successively take on the colors blue, green and red.



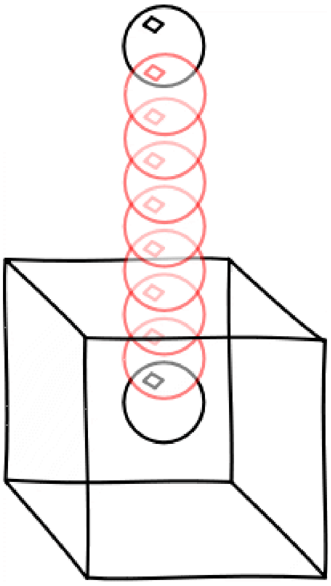
Now let's apply this colored layer trick to the earlier example of lifting a coin out of a frame. The coin starts in the same two dimensional space as the frame. We lift it up into the third dimension into a higher spatial layer that we have color-coded red. In this higher layer, the coin can move freely left/right and front/back without intersecting the frame. We moving it to the right until it passes over the frame. Then we lower it back down outside.

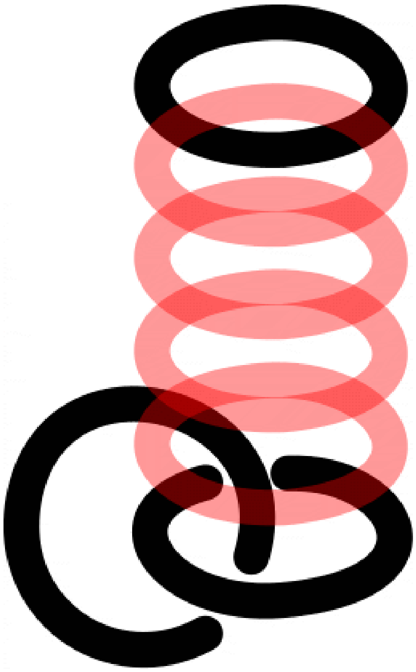


Now imagine that we cannot perceive the third dimension directly. Here's how we'd picture the coin's escape. It starts out inside the frame in the space of the frame. It is then lifted out of the frame into the third dimension. At that moment, it is indicated by a ghostly red coin. Its spatial position in the left/right and front/back direction has not changed. All that has changed is its height. It is now in the red height layer. If we move the coin left or right, or front and back, in this red layer, it no longer intersects the frame and can move right over it. We won't see it move over the frame, however. As far as we are concerned it will just move through it.

The motion of the coin in this third dimensional escape passage is illustrated by the ghostly red coin.

This last analysis of the coin in the frame is the template for dealing with the real case of a marble trapped inside a three dimensional box. If the marble moves in any of the three familiar dimensions (up/down, left/right and front/back), its motion intersects the walls of the box and it cannot escape. So we lift the marble into the fourth dimension, without changing its position in the three familiar dimensions. In the figure, this is shown by the marble turning ghostly red. In the red space, the marble is free to move up/down, left/right and front/back, without intersecting the box's walls. The marble then moves so that is passes over one of the walls. It is then lowered out of the red space back to the original three dimensional space of the box, but now outside the walls.





The same analysis applies to the linked rings. One ring is lifted out of the three dimensional space of the original set up. In this red space, the ring can move freely without intersecting the other ring. We move it well away from the other ring and then drop it back into the original three dimensional space. It is now unlinked from the other ring.

What you should know

- The properties of squares, cubes and tesseracts.
- How to arrive at the properties of a tesseract and other four-dimensional figures by extrapolating the methods used to get the properties of a cube.